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The Method of Fundamental Solutions for the Solution of Steady-State Free Boundary Problems

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The method of fundamental solutions is formulated for the solution of steady-state free boundary problems. The method is tested on three such problems and the results compared to published results obtained with other numerical methods. © 1992 Academic Press, Inc.

1. INTRODUCTION

In recent years the method of fundamental solutions (MFS) has proved to be an effective alternative to boundary element methods for the numerical solution of certain elliptic boundary value problems. The first applications of the MFS were to two- and three-dimensional linear potential problems (Mathon and Johnston [17], Fairweather and Johnston [5], and Johnston and Fairweather [7]). Subsequently the method was applied very effectively to nonlinear plane potential problems (Karageorghis and Fairweather [13]), to problems involving the modified Laplace/Helmholtz equation (Johnston and Mathon [8], MacDonell [16]) and to biharmonic problems (Karageorghis and Fairweather [10–12]).

Traditional boundary methods, such as the boundary integral equation method, have been used extensively for the numerical solution of steady-state free boundary problems (Liggett [15], Kelmanson [14], Aitchison and Karageorghis [3], Karageorghis [9], Aitchison and Karageorghis [4]). These methods deal directly with the boundary of the region of the problem under consideration and thus have an advantage over domain discretization methods, such as finite element or finite difference methods. Since in free boundary problems it is the boundary of the region which is of prime interest, boundary methods are ideally suited for the numerical solution of such problems. Further, the determination of the position of the free boundary introduces variables which appear nonlinearly in the global system of equations, irrespective of the numerical method used. One is therefore required to devise iterative schemes for the solution of the problem. As well as offering the advantages of boundary methods over domain discretization methods for the solution of free boundary problems, the MFS can absorb these nonlinearities naturally at little extra effort, thus avoiding the need to design an appropriate iterative scheme.

In Section 2, the MFS is formulated for the solution of free boundary problems. Section 3 describes the details involved in the application of the method to such problems and examines a problem with a known analytical solution. Sections 4, 5, and 6 present applications of the method to three problems from the literature. As is often the case with free boundary problems each of these has different features and the flexibility of the method is highlighted in each application.

2. FORMULATION OF THE MFS FOR SOLVING FREE BOUNDARY PROBLEMS

The MFS is first described for the solution of a fixed boundary problem governed by Laplace's equation. Following Mathon and Johnston [17], the following problem is considered

$$\nabla^2\phi(\mathbf{p}) = 0 \qquad \text{for} \quad \mathbf{p} \in \Omega, \tag{2.1}$$

subject to the boundary condition:

$$B\phi(\mathbf{p}) = 0 \qquad \text{for} \quad \mathbf{p} \in \partial\Omega. \tag{2.2}$$

The solution to the boundary value problem (2.1)–(2.2) is approximated by

$$\phi_N(\mathbf{c}, \mathbf{t}, \mathbf{p}_i) = \sum_{j=1}^{N} c_j k(\mathbf{t}_j, \mathbf{p}_i), \tag{2.3}$$

where

$$\mathbf{c} = [c_1, c_2, c_3, ..., c_N]^{\mathrm{T}},$$

$$\begin{aligned} k(\mathbf{t}_j, \mathbf{p}_i) &= \log r_{ji}, \\ r_{ji} &= [(t_{j_1} - p_{i_1})^2 + (t_{j_2} - p_{i_2})^2]^{1/2} \\ \mathbf{t} &= [t_{1_1}, t_{1_2}, t_{2_1}, t_{2_2}, ..., t_{N_1}, t_{N_2}]^{\mathrm{T}}. \end{aligned} \tag{2.4}$$

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The pair (t_{j_1}, t_{j_2}) represents the coordinates of the singularity $\mathbf{t}_j$ (a point outside $\bar{\Omega}$) and the pair (p_{i_1}, p_{i_2}) are the coordinates of the point $\mathbf{p}_i (\mathbf{p}_i \in \bar{\Omega})$. The solution to (2.1)–(2.2) is thus approximated by a linear combination of fundamental solutions of Laplace's equation. The coefficients c_j and the positions of the singularities $\mathbf{t}_j$ are chosen so that ϕ_N satisfies the boundary condition (2.2) as well as possible, in a least squares sense. This is done by choosing M points $\mathbf{p}_i$ on the boundary $\partial\Omega$ and subsequently minimizing the functional

$$F(\mathbf{c}, \mathbf{t}) = \sum_{i=1}^{M} |B\phi_N(\mathbf{c}, \mathbf{t}, \mathbf{p}_i)|^2, \tag{2.5}$$

which is nonlinear in the $\mathbf{t}_j$, using a nonlinear least squares algorithm.

In the case of a free boundary problem, one considers Laplace's equation

$$\nabla^2\phi(\mathbf{p}) = 0 \qquad \text{for} \quad \mathbf{p} \in \Omega, \tag{2.6}$$

subject to the boundary conditions

$$B\phi(\mathbf{p}) = 0 \qquad \text{for} \quad \mathbf{p} \in \partial\Omega_{\text{FIXED}} \tag{2.7}$$

and

$$\left.\begin{aligned} B_1\phi(\tilde{\mathbf{p}}) = 0 \\ B_2\phi(\tilde{\mathbf{p}}) = 0 \end{aligned}\right\} \qquad \text{for} \quad \tilde{\mathbf{p}} \in \partial\Omega_{\text{FREE}}, \tag{2.8}$$

where $\partial\Omega = \partial\Omega_{\text{FIXED}} \cup \partial\Omega_{\text{FREE}}$, $\partial\Omega_{\text{FIXED}}$ being the fixed part of the boundary and $\partial\Omega_{\text{FREE}}$ being the free part of the boundary, the geometry of which is unknown. If M_{FIXED} and M_{FREE} boundary points are placed on the fixed and free parts of the boundary, respectively, it is now necessary to minimize the functional

$$\begin{aligned} F(\mathbf{c}, \mathbf{t}, \tilde{\mathbf{p}}) = & \sum_{i=1}^{M_{\text{FIXED}}} |B\phi_N(\mathbf{c}, \mathbf{t}, \mathbf{p}_i)|^2 \\ & + \sum_{i=1}^{M_{\text{FREE}}} \{|B_1\phi_N(\mathbf{c}, \mathbf{t}, \tilde{\mathbf{p}}_i)|^2 \\ & + |B_2\phi_N(\mathbf{c}, \mathbf{t}, \tilde{\mathbf{p}}_i)|^2\} \end{aligned} \tag{2.9}$$

which is now also nonlinear in the coordinates of the free boundary points $\tilde{\mathbf{p}}$, $\tilde{\mathbf{p}}_i = (\tilde{p}_{i_1}, \tilde{p}_{i_2})$, $i = 1, 2, ..., M_{\text{FREE}}$. In most practical applications it is sufficient to allow only a subset of the coordinates of the free boundary points to move; e.g., one may choose to allow only the y-coordinates to move while keeping the x-coordinates fixed and vice versa.

A point of practical importance arises when the boundary conditions on the free boundary involve the outward normal derivative $\partial/\partial n$ of ϕ_N. In order to obtain this one requires the knowledge of the outward unit normal vector $\mathbf{n}$ at each free boundary point $\tilde{\mathbf{p}}_i$, $i = 1, 2, ..., M_{\text{FREE}}$. For a curve $y = g(x)$ the outward unit normal vector is given by

$$\mathbf{n} = \frac{(\pm g'(x), \pm 1)}{\sqrt{1 + g'(x)^2}}. \tag{2.10}$$

For each set of free boundary points we update $\mathbf{n}$ by using a central difference approximation:

$$g'(\tilde{p}_{i_1}) = \frac{\tilde{p}_{i+1_2} - \tilde{p}_{i-1_2}}{\tilde{p}_{i+1_1} - \tilde{p}_{i-1_1}}, \qquad i = 2, ..., M_{\text{FREE}} - 1. \tag{2.11}$$

Also, $g'(\tilde{p}_{1_1})$ and $g'(\tilde{p}_{M_{\text{FREE}_1}})$ may be approximated by one-sided finite difference approximations.

3. APPLICATION OF THE MFS

As in previous applications of the MFS (e.g., [12]) the least squares algorithm used is the MINPACK routine LMDIF, which implements a modified version of the Levenberg–Marquardt algorithm.

The routine LMDIF minimizes the sum of squares

$$F = \sum_{i=1}^{M} |f_i|^2, \tag{3.1}$$

where in the present application

$$\begin{aligned} f_i &= \sum_{j=1}^{N} c_j Bk(\mathbf{t}_j, \mathbf{p}_i), \\ &\qquad i = 1, ..., M_{\text{FIXED}} \\ f_i &= \sum_{j=1}^{N} c_j B_1 k(\mathbf{t}_j, \tilde{\mathbf{p}}_i), \\ &\qquad i = M_{\text{FIXED}} + 1, ..., M_{\text{FIXED}} + M_{\text{FREE}} \\ f_i &= \sum_{j=1}^{N} c_j B_2 k(\mathbf{t}_j, \tilde{\mathbf{p}}_i), \\ &\qquad i = M_{\text{FIXED}} + M_{\text{FREE}} + 1, ..., M, \end{aligned} \tag{3.2}$$

where $M = M_{\text{FIXED}} + 2\,M_{\text{FREE}}$. In this routine one is not required to provide the Jacobian, which is approximated internally by a finite difference scheme. LMDIF terminates when either the user-specified tolerance is achieved or the user-specified limit on the number of function evaluations is reached.

An important factor in the least squares procedure is the initial placement of the singularities. In the fixed boundary case, an efficient rule is to distribute both the boundary points and the starting singularities, uniformly around the boundary. The singularities are initially placed at a fixed (user-specified) distance along the normals to the surface (for details see Ho-Tai *et al.* [6] or Karageorghis and Fairweather [9]). The starting values of the coefficients c_j are all taken to be equal to the same constant (usually unity).

In the solution of free boundary problems, the same rules are applied to the fixed part of the boundary. A denser distribution of both boundary points and singularities is usually taken in the region of the free boundary. The initial distance of the singularities from the boundary in the region of the free boundary is taken to be shorter than for singularities corresponding to the fixed part of the boundary. The initial shape of the free boundary depends to a great extent on the particular problem under consideration. In each of the cases examined in this study the free boundary is initially set to be a straight line. Further details concerning this choice accompany each example.

One of the drawbacks of the method is the tendency of the singularities to move to the interior of the domain of the problem under consideration. In the present implementation an internal check is added to the subroutine of LMDIF which evaluates the functions f_i. In it, the positions of the singularities are checked with respect to an approximation to the domain of the problem (since the exact domain is not known). If a singularity lies inside the domain it is removed and repositioned at the exterior of the domain.

Before examining the application of the method to physical problems, the MFS is tested on a problem which is free of singularities and for which an analytical solution is known. This is done in order to investigate the convergence of the method as M, N, and the number of function evaluations (NFEV) are increased.

The problem examined has the exact solution $\phi(x, y) = 0.5 + y$ and its free boundary is given by $h(x) = 0.5 + a\cos\pi(x - 0.5)$. The problem is solved in $-0.5 \leqslant x \leqslant 0.5$ for $y \geqslant -0.5$. The boundary conditions are

$$\frac{\partial\phi}{\partial x} = 0 \quad \text{on} \quad x = -0.5,\ -0.5 \leqslant y \leqslant h(-0.5),$$

$$\frac{\partial\phi}{\partial x} = 0 \quad \text{on} \quad x = 0.5,\ -0.5 \leqslant y \leqslant h(0.5),$$

$$\phi = 0 \quad \text{on} \quad y = -0.5,\ -0.5 \leqslant x \leqslant 0.5$$

and

$$\left.\begin{aligned} &\phi = 0.5 + a\cos\pi(x - 0.5), \\ &\frac{\partial\phi}{\partial n} = (1 + a^2\pi^2\sin^2\pi(x - 0.5))^{-1/2} \end{aligned}\right\} \text{ on the free boundary.}$$

The unknowns of the problem are the solution $\phi(x)$ and the shape of the free boundary $h(x)$. An initial guess for $h(x)$ is taken to be a straight line joining the points $(-0.5, h(0.5))$ and $(0.5, h(0.5))$, where $h(-0.5)$ and $h(0.5)$ are unknown (and therefore given appropriate starting values). The regularity of the solution forces the free boundary to be horizontal at the points $(-0.5, h(-0.5))$ and $(0.5, h(0.5))$, and these conditions are enforced numerically. The initial boundary is divided into equally spaced boundary points and the singularities are initially spaced uniformly around the boundary. The y-coordinates of the free boundary (including the end points) are allowed to move.

Numerical experiments were carried out for three values of the amplitude a and various numbers of boundary points, singularities and function evaluations. Since we seek to establish the convergence of the method, a sufficiently large number of function evaluations was used for each case, in order to achieve numerical convergence. In Tables I(a)–(c), we give the number of function evaluations, number of boundary points, number of singularities, and the maximum error in the height of the free boundary at the free boundary points, for $a = 0$, 0.05, and 0.2, respectively. For $a = 0$, the exact solution is a straight line ($y = 0.5$) and it appears that very few degrees of freedom are sufficient for its accurate representation. For $a = 0.05$, 0.2, the problem becomes harder and more degrees of freedom are required to represent the free boundary accurately. In particular, 42

TABLE I

MFS Results for Test Example with Exact Solution

	NFEV	Boundary points (N)		
(a) $a = 0$		42(4)	62(6)	82(8)
	1000	0.9855(−4)	0.1399(−3)	0.2963(−5)
	2000	0.3848(−4)	0.2334(−4)	0.9393(−7)
	3000	0.2114(−4)	0.1112(−4)	0.8001(−7)
	4000	0.1346(−4)	0.6308(−5)	0.7280(−7)
	5000	0.9755(−5)	0.3975(−5)	0.6768(−7)
	6000	0.6913(−5)	0.2727(−5)	0.6267(−7)
(b) $a = 0.05$		42(4)	82(8)	122(12)
	1000	0.6731(−3)	0.1770(−3)	0.3377(−3)
	2000	0.2667(−3)	0.9691(−4)	0.1533(−3)
	3000	0.2242(−3)	0.7901(−4)	0.7286(−4)
	4000	0.2274(−3)	0.7145(−4)	0.4499(−4)
	5000	0.2287(−3)	0.6721(−4)	0.3478(−4)
	6000	0.2303(−3)	0.6495(−4)	0.3138(−4)
(c) $a = 0.2$		42(4)	82(8)	122(12)
	1000	0.3022(−2)	0.1346(−2)	0.7989(−3)
	2000	0.2635(−2)	0.9199(−3)	0.4896(−3)
	3000	0.2604(−2)	0.8244(−3)	0.3989(−3)
	4000	0.2610(−2)	0.7804(−3)	0.3645(−3)
	5000	0.2612(−2)	0.7515(−3)	0.3425(−3)
	6000	0.2613(−2)	0.7340(−3)	0.3290(−3)

boundary points are clearly not sufficient, with the error stabilizing at a relatively large value.

All computations were performed in double precision on an IBM 3081–D32 computer located at Southern Methodist University.

4. EXAMPLE 1: SEEPAGE THROUGH A RECTANGULAR POROUS DAM

In the first example the MFS is applied to the problem of free surface flow through a rectangular block of soil depicted in Fig. 1. Water seeps slowly from the left (AB) to the right (DC) through the dam. In this problem, the governing equations in terms of the potential function $\phi(x, y)$ are

$$\nabla^2\phi = 0 \quad \text{in } ABCDEA, \tag{4.1}$$

subject to the boundary conditions

$$\phi = H - h \quad \text{on } AB, \tag{4.2}$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on } BC, \tag{4.3}$$

$$\phi = 0 \quad \text{on } CD, \tag{4.4}$$

$$\phi = y \quad \text{on } DE, \tag{4.5}$$

and

$$\left.\begin{aligned} \phi &= y \quad (4.6)\\ \frac{\partial\phi}{\partial n} &= 0 \quad (4.7)\end{aligned}\right\} \quad \text{on } AE.$$

The unknowns in this problem are the potential function $\phi(x, y)$, the shape of the free boundary AE, and the height h_0 (CE).

The MFS is applied to the case when $H = 10$, $h = 3$, and $L = 10$. The free boundary is initially taken to be a straight line through the point A, intersecting the line CD at the point E_0, where $h < CE_0 < H$. In order to determine the free boundary AE, the x-coordinates of the free boundary points are kept fixed, while the y-coordinates (including the boundary point E) are allowed to move. The y-coordinates of the boundary points on DE are readjusted (in LMDIF) for each value of the unknown h_0. The problem has a boundary singularity at E and therefore in the neighbourhood of E a denser grid is imposed by placing the x-coordinates of the free boundary points at a distance

$$d_j = L\left(\frac{j}{M_{\text{FREE}}}\right)^2, \qquad j = 1, 2, \ldots \tag{4.8}$$

from the vertical line CF. Similarly a denser grid is placed on DE.

A sample of results obtained using the MFS is listed in Table II. The height h_0 which locates the free surface-seepage intersection is found to be in good agreement with the analytical solution of Polubarinova-Kochina [19] ($h_0 \approx 4.20$) and the results of Liggett [15] obtained using a boundary integral equation method. Also listed in Table I are the number of function evaluations (NFEV) and the corresponding CPU times required for each computation.

Further, Fig. 2a–e show the locations of the singularities and of the free surface from the starting guess to their location after 1000 function evaluations, for the case when $M = 98$, $N = 10$, $M_{\text{FREE}} = 20$, in steps of 250 function evaluations. Finally Fig. 2f shows the positions of the free boundary and singularities for much fewer degrees of freedom.

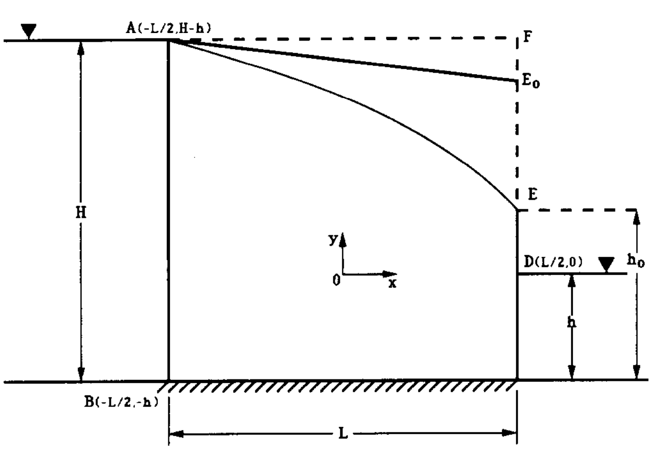

FIG. 1. Seepage through a rectangular porous dam.

TABLE II

MFS Results for Example 1

M	M_{FREE}	N	h_0	NFEV	CPU(s)
68	10	7	4.254	400	12.20
68	10	7	4.147	600	17.60
68	10	7	4.212	800	23.31
68	10	7	4.223	1000	29.64
98	20	10	4.245	500	31.80
98	20	10	4.270	750	47.60
98	20	10	4.260	1000	61.20
98	20	10	4.269	1250	74.66
128	30	13	4.193	500	54.67
128	30	13	4.226	750	85.54
128	30	13	4.273	1000	113.58
128	30	13	4.273	1500	156.38

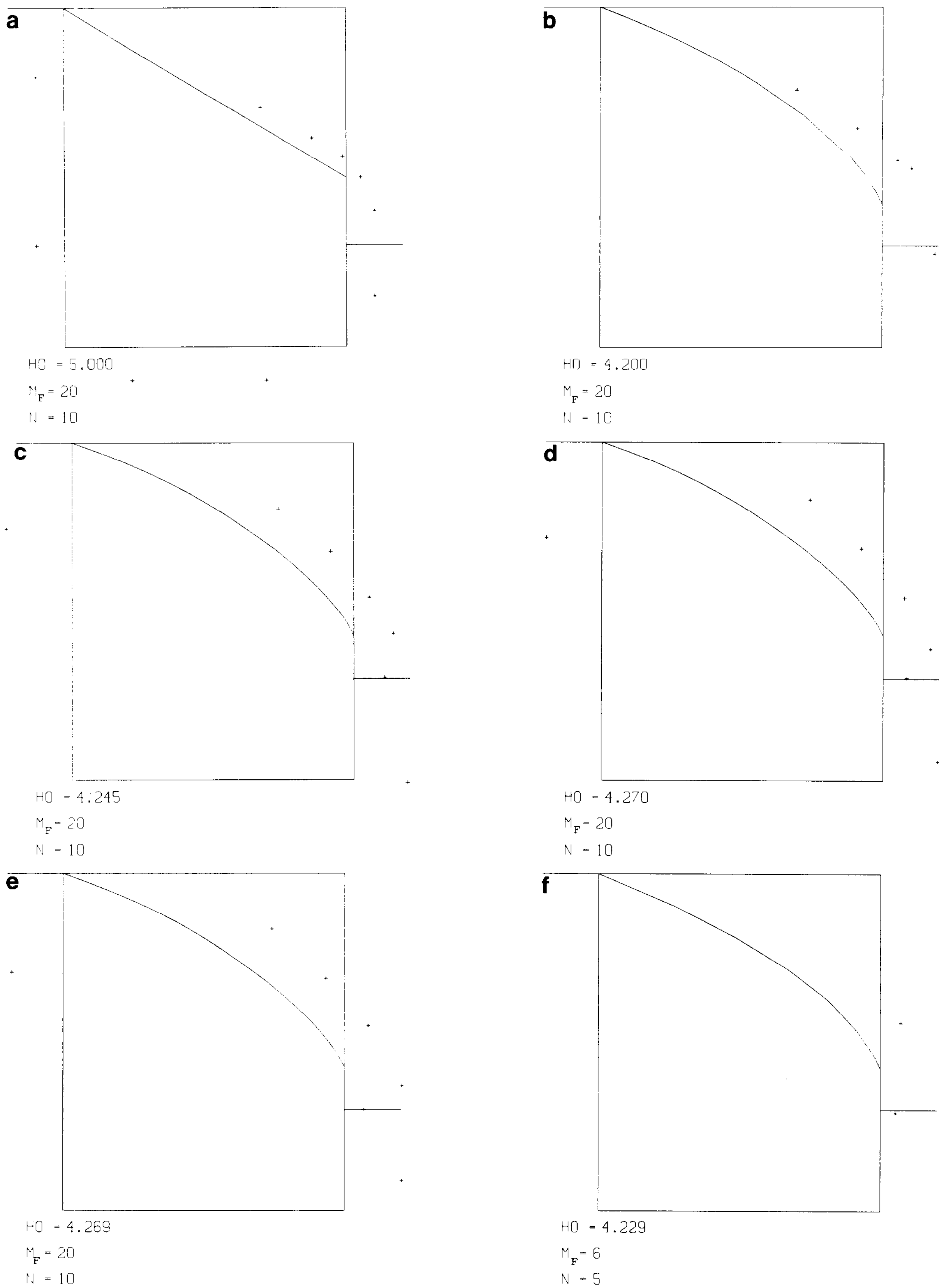

FIG. 2. Free boundary for Example 1 with $M = 98$, $N = 10$, $M_{\text{FREE}} = 20$: (a) Initial setting; (b) NFEV = 250; (c) NFEV = 500; (d) NFEV = 750; (e) NFEV = 1000; (f) $M = 60$, $M_{\text{FREE}} = 6$, $N = 5$, NFEV = 200.

5. EXAMPLE 2. RIABOUCHINSKY CAVITY FLOW

This example examines the planar flow of an incompressible inviscid fluid past a plate in a channel of finite width and infinite length. The plate is placed symmetrically in the channel at right angles to the flow. The aim is to find the shape of the cavity formed behind the plate.

One may consider a model by Riabouchinsky [20], depicted in Fig. 3, in which the cavity is closed by introducing an image plate downstream of the plate. By exploiting the symmetry possessed by the flow about the axis of the channel and the vertical line through the midpoint of the line joining the centres of the two plates, only a quarter of the region needs to be considered (see Fig. 4). The model can be non-dimensionalised (see Aitchison [2]) leading to the following problem for the stream function ψ in the region $ABCDEFA$ (Fig. 4),

$$\nabla^2\psi = 0 \qquad \text{in } ABCDEFA, \tag{5.1}$$

subject to the boundary conditions

$$\frac{\partial\psi}{\partial n} = 0 \qquad \text{on } CD, AB \tag{5.2}$$

$$\psi = 0 \qquad \text{on } DE, EF \tag{5.3}$$

$$\psi = 1 \qquad \text{on } BC \tag{5.4}$$

and

$$\left.\begin{aligned} \psi &= 0 && (5.5)\\ \frac{\partial\psi}{\partial n} &= -q && (5.6)\end{aligned}\right\} \quad \text{on } FA,$$

where the unknowns in this problem are the stream function $\psi(x, y)$, the shape of the free boundary FA, the height of the point A (b), and the constant q.

The MFS is applied to the problem with $L = 0.5$, $CD = 1$ ($h = 1.0$), $DE = 1.5$, $d = 0.1$. The free boundary is initially taken to be a straight line through the point F, intersecting the line AB at the point A_0, where $h - d > A_0B > h/2$. To determine the free boundary FA, the x-coordinates of the free boundary points are kept fixed and their y-coordinates allowed freedom of movement (including the point A). As in the previous example the y-coordinates of the boundary AB are readjusted for each value of the y-coordinate of A. The initial value of q is taken to be equal to zero.

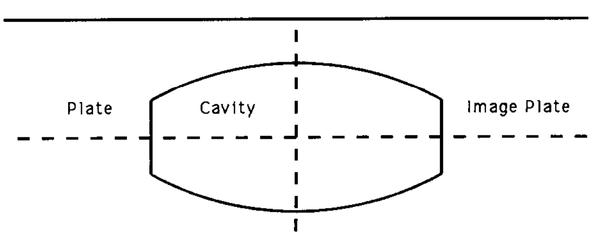

FIG. 3. Riabouchinsky cavity model.

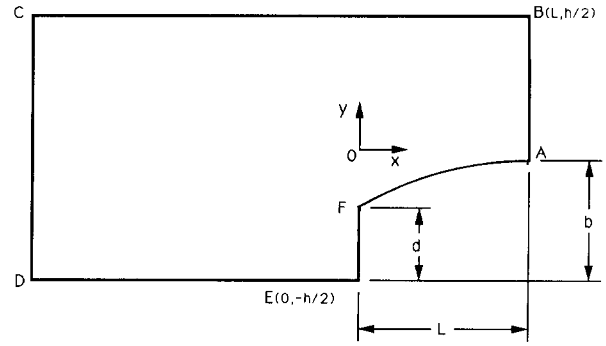

FIG. 4. Region of solution for Example 2.

The problem has a boundary singularity at the point F. Account of this singularity may be taken by exploiting knowledge of the shape of the free boundary in the neighbourhood of F (Aitchison [2]), where in terms of local coordinates about F,

$$\xi = a\eta^{2/3} \qquad \left(\eta = x,\ \xi = y + \frac{h}{2} - d\right). \tag{5.7}$$

TABLE III

MFS Results for Example 2

M	M_{FREE}	N	q	b	NFEV	CPU(s)
132	15	12	1.4491	0.2127	1000	83
132	15	12	1.4723	0.2200	2000	160
132	15	12	1.4963	0.2306	3000	235
132	15	12	1.5154	0.2386	4000	319
132	15	12	1.5219	0.2418	5000	395
132	15	12	1.5199	0.2409	6000	463
142	20	14	1.4429	0.2297	1000	116
142	20	14	1.4880	0.2276	2000	218
142	20	14	1.4899	0.2284	3000	307
142	20	14	1.4943	0.2302	4000	410
142	20	14	1.4992	0.2323	5000	501
142	20	14	1.5064	0.2354	6000	598
162	30	15	1.4906	0.2867	1000	146
162	30	15	1.5421	0.2584	2000	268
162	30	15	1.5357	0.2489	3000	401
162	30	15	1.5310	0.2473	4000	524
162	30	15	1.5305	0.2470	5000	656
162	30	15	1.5302	0.2469	6000	777
Mogel and Street [18]			1.562	0.245		
Aitchison [2]			1.429	0.2013		
Aitchison and Karageorghis [4]			1.4753	0.2222		

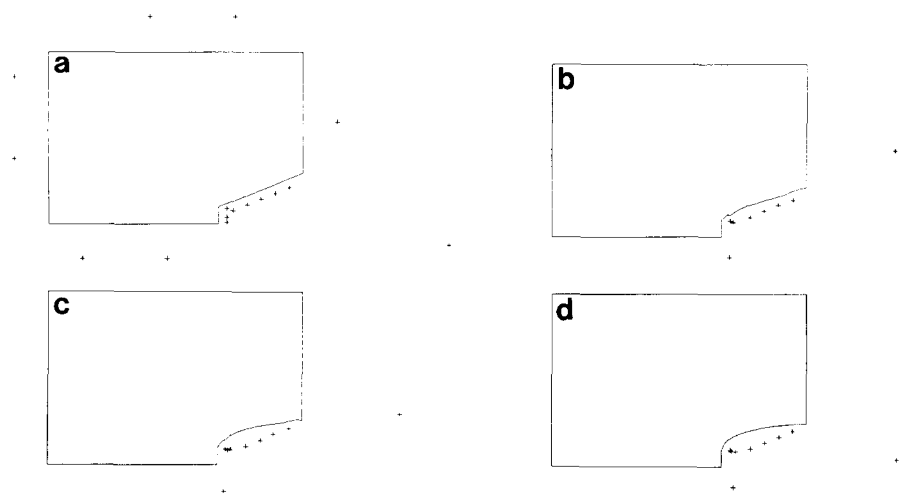

FIG. 5. Location of free boundary and singularities for Example 2 with $M = 162$, $M_{\mathrm{FREE}} = 30$, $N = 15$: (a) initial setting; (b) NFEV = 1000; (c) NFEV = 2000; (d) NFEV = 3000.

If this behaviour is assumed at the first boundary point, clearly $a = \xi_1/\eta_1^{2/3}$. It is therefore possible to use this known behaviour of the free boundary to derive the unit normal **n** at the first (or first few) boundary point(s) instead of using the finite difference scheme described in Section 2. In most numerical experiments this behaviour is imposed only at the first free boundary point. At the free boundary point A, the normal to the free boundary is assumed to be vertical.

A set of results obtained using the MFS is listed in Table III for three different discretizations of the problem. For each discretization, results for the constant value q of the normal derivative along the free boundary and the height b are listed for various numbers of function evaluations (NFEV) and the corresponding CPU times required for each computation. For each set of degrees of freedom, the values of q and b obtained are in close agreement with the corresponding values of Mogel and Street [18] obtained using a finite difference method, Aitchison [2] obtained using a finite element method and Aitchison and Karageorghis [4] obtained using a boundary integral equation method.

The initial positions of the singularities and the initial location of the free boundary are shown in Fig. 5a for the case when $M = 162$, $M_{\mathrm{FREE}} = 30$, $N = 15$. Figures 5b–d show the successive positions of the singularities and locations of the free boundary for NFEV = 1000, 2000, and 3000. Magnified plots of the free boundary for three sets

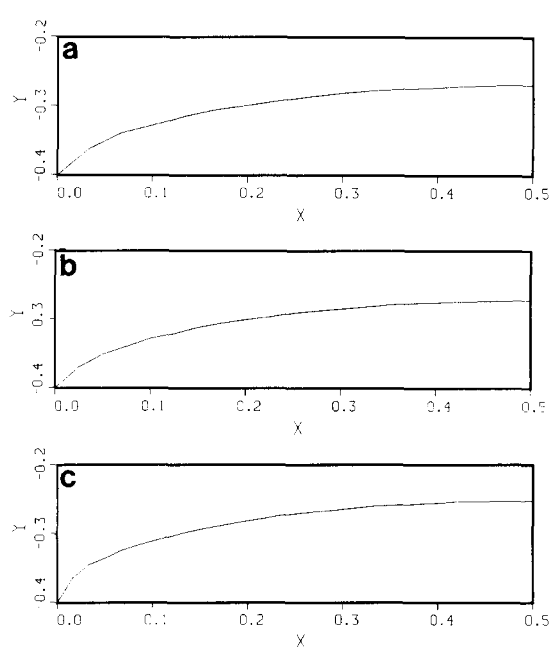

FIG. 6. Magnification of free boundary for Example 2: (a) $M = 132$, $M_{\mathrm{FREE}} = 15$, $N = 12$; (b) $M = 142$, $M_{\mathrm{FREE}} = 20$, $N = 14$; (c) $M = 162$, $M_{\mathrm{FREE}} = 30$, $N = 15$.

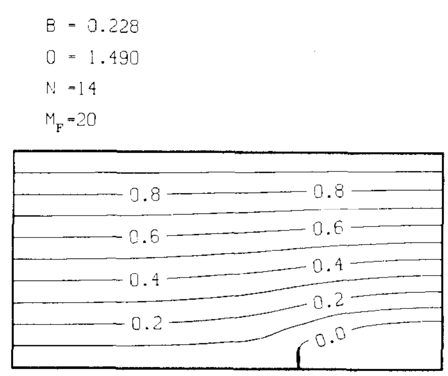

FIG. 7. Stream function contours for Example 2 with $M = 142$, $M_{\mathrm{FREE}} = 20$, $N = 14$.

of discretizations are presented in Fig. 6. From Fig. 6c it is evident that the method captures the correct behaviour of the free boundary near the point F more accurately as the number of boundary points M_{FREE} is increased. A contour plot of the stream function is depicted in Fig. 7, for $M = 142$, $M_{\mathrm{FREE}} = 20$, $N = 14$.

6. EXAMPLE 3: FLOW OVER A WEIR

The final example examines the potential flow over a triangular weir under gravity (Fig. 8). The stream function $\psi(x, y)$ satisfies

$$\nabla^2\psi = 0 \quad \text{in } ABCDEFGA \tag{6.1}$$

subject to the boundary conditions

$$\psi = 0 \quad \text{on } ABCDE \tag{6.2}$$

$$\frac{\partial\psi}{\partial n} = 0 \quad \text{on } EF, GA \tag{6.3}$$

and

$$\left.\begin{aligned} \psi &= q && (6.4)\\ \frac{\partial\psi}{\partial n} &= \sqrt{2g(H_0 - y - 0.5)} && (6.5)\end{aligned}\right\} \text{ on } FG,$$

where q is the discharge and H_0 defines the stagnation level.

For a given value of H_0 there are three physically interesting solutions to this problem. Following the analysis of Aitchison [1], by assuming fully developed flow on EF and GA, the heights of $EF(h_2)$, and $GA(h_1)$ must satisfy (given q) the cubic

$$z^3 - H_0 z^2 + \frac{q^2}{2g} = 0. \tag{6.6}$$

TABLE IV

Values of q with Number of Function Evaluations NFEV for $M = 197$, $M_{\mathrm{FREE}} = 45$, $N = 12$

q	NFEV	q	NFEV
1.0000	80	3.3472	805
2.0756	160	3.3440	885
3.2301	241	3.3409	965
3.3242	322	3.3393	1045
3.2513	403	3.3386	1125
3.2511	483	3.3383	1205
3.3909	563	3.3382	1285
3.3476	644	3.3381	1365
3.3528	724		

Equation (6.6) has two real positive roots z_1, z_2 in $[0, H_0]$ (and a negative one which is of no interest), provided

$$\frac{q^2}{g} < \frac{3H_0^3}{27}. \tag{6.7}$$

If inequality (6.7) is satisfied, $z_1 \in [0, 2H_0/3]$ and $z_2 \in [2H_0/3, H_0]$. The physically interesting flows are defined by

$$h_1 = h_2 = z_1 \quad \text{(supercritical flow)} \tag{6.8}$$

$$h_1 = h_2 = z_2 \quad \text{(subcritical flow)} \tag{6.9}$$

$$h_1 = z_2, \quad h_2 = z_1 \quad \text{(critical flow).} \tag{6.10}$$

Of particular interest is the case of critical flow, in which one needs to specify either H_0 or q, the remaining constant being part of the solution. In the present application H_0 is prescribed and the unknowns in the boundary value problem (6.1)–(6.5) are the stream function ψ, the discharge q, and the shape of the free boundary FG.

The MFS is applied to the particular case when $XR = XL = 3.0$, $p = 0.2$, and $H_0 = 1.138$ (see Fig. 8). Initially, the unknown constant q is set equal to 1. The heights h_1 and h_2 are found by solving (6.6) and setting $h_1 = z_2$, $h_2 = z_1$. The

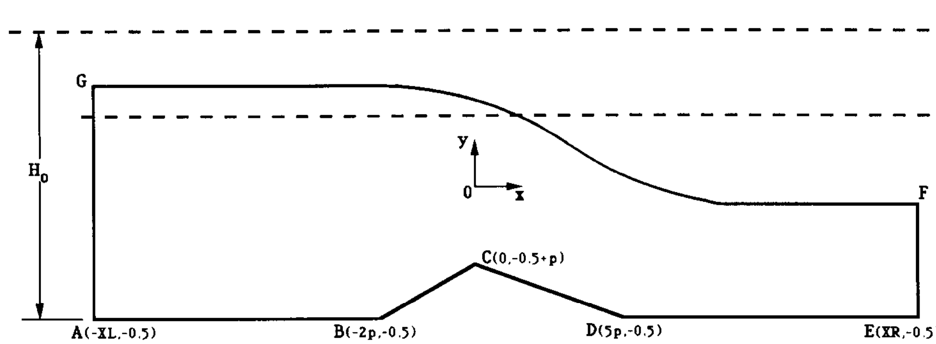

FIG. 8. Flow over a triangular weir.

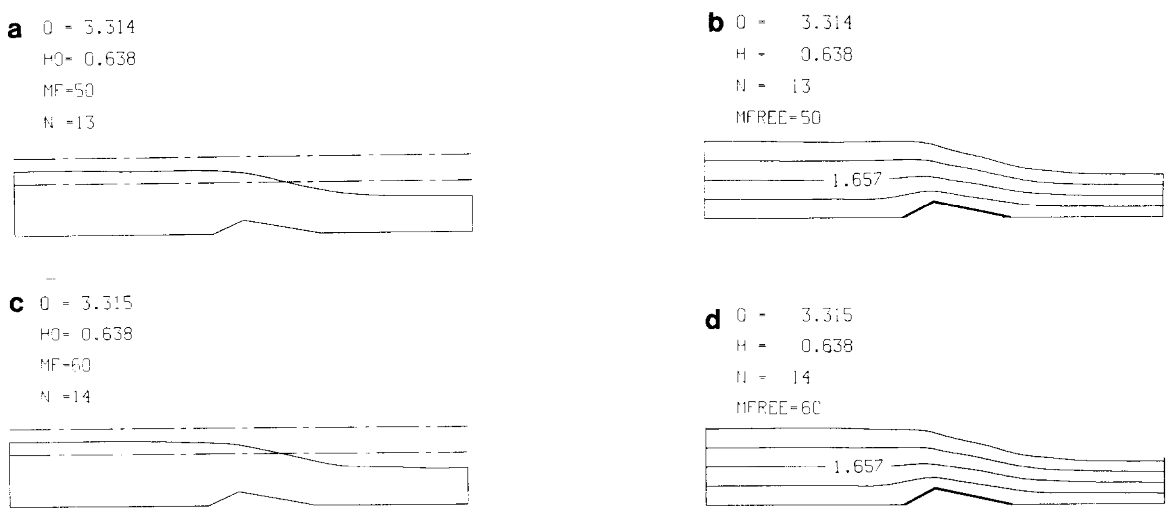

FIG. 9. (a) Shape of free boundary for Example 3 with $M=177$, $M_{\text{FREE}}=50$, $N=13$, NFEV $=2500$; (b) stream function contours for $M=177$, $M_{\text{FREE}}=50$, $N=13$, NFEV $=2500$; (c) shape of free boundary for Example 3 with $M=227$, $M_{\text{FREE}}=60$, $N=14$, NFEV $=4000$; (d) stream function contours for $M=227$, $M_{\text{FREE}}=60$, $N=14$, NFEV $=4000$.

shape of the free boundary is thus initially taken to be a straight line through the points G and F. Subsequently, for each value of the constant q, the heights h_1 and h_2 are updated by solving (6.6). Similarly the y-coordinates of the boundary points of EF and GA are updated for each value of q. The free parameters of the free boundary are taken to be the y-coordinates of the free boundary points.

Because of the unusual shape of the region and the length of the free boundary, for the satisfactory solution of the problem more degrees of freedom are required than in either of the preceding examples. In all cases the method converges to a free boundary with no waves and to a value of q close to 3.30 which is the critical value of q obtained by Aitchison [1]. A typical example at the convergence of q is given in Table IV.

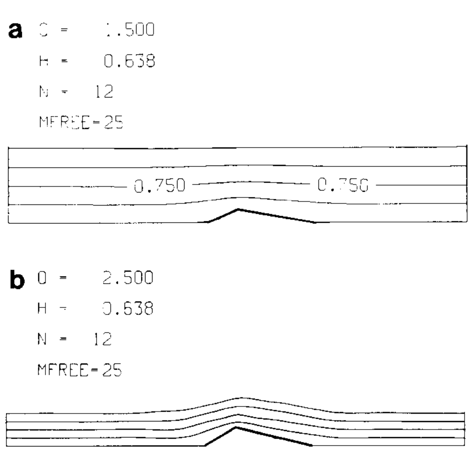

FIG. 10. (a) Subcritical flow for $q=1.5$ with $M=137$, $M_{\text{FREE}}=25$, $N=12$, NFEV $=2000$; (b) supercritical flow for $q=2.5$ with $M=137$, $M_{\text{FREE}}=25$, $N=12$, NFEF $=2000$.

The shape of the free boundary and a stream function contour plot after 2500 function evaluations for the case $M=177$, $M_{\text{FREE}}=50$, $N=13$ are presented in Fig. 9a, b. In this case q converges to 3.314. Similarly, Fig. 9c, d present the case $M=227$, $M_{\text{FREE}}=60$, $N=14$ for 4000 function evaluations and the value of q converging to 3.315.

The cases of subcritical flow and supercritical flow are considerably easier. For these flows, both q (and hence h_1 or h_2) and H_0 are prescribed in advance. Figure 10a shows the stream function contour for the case of subcritical flow for $q=1.5$ ($H_0=1.138$) for $M=137$, $M_{\text{FREE}}=25$, $N=12$. Figure 10b shows the case of supercritical flow for $q=2.5$ for the same discretization.

7. CONCLUSIONS

In this study, the performance of the MFS is examined on three free boundary problems. The ease of implementation of the method and the fact that it only deals with the boundary of the region of the problem under consideration make it well suited for the numerical solution of the problems in question. The parameters describing the shape and position of the free boundary are naturally included in the non-linear least squares minimization scheme, thus avoiding the necessity to devise iterative schemes for repositioning the free surface, the convergence of which could be doubtful. Further, the method can also accommodate extra unknowns of the problem (Examples 2 and 3) at no extra effort. In almost all the cases examined the method converges to the correct solution with poor initial approximations for the position of the free boundary and other unknown quantities. In some instances, due to the presence of boundary singularities, one expects a relatively large number of degrees of freedom to accurately represent the solution. This situation was observed in Example 2 (Table II), where poor convergence is obtained for few degrees of freedom. It is worth noticing the accumulation of sources near the singular point (Fig. 5) which confirms that the method captures the qualitative behaviour of the solution. One of the drawbacks of the method is the tendency of the singularities to move to the interior of the domain. This can be easily overcome with an internal check as described in Section 3.

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